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Number-difference-phase coherent state analogue in two-mode Fock space

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Abstract. We propose a new number-difference-phase coherent state analogue in two-mode Fock space by introducing a new operator $A = \sqrt{a^\dagger a - b^\dagger b + 1}(\frac{a+b^\dagger}{a^\dagger+b})^{\frac{1}{2}}$. The coherent state analogue is the eigenvector of A and possesses non-orthonormal and overcompleteness properties. It is constructed on certain superposition states in the radius direction.

1. Introduction

It is well known [1] that the coherent state is the eigenvector of the photon annihilation operator a, i.e. $a|\alpha\rangle = \alpha|\alpha\rangle$, where $|\alpha\rangle = \exp[-\frac{1}{2}|\alpha|^2 + \alpha a^\dagger]|0\rangle$ and possesses the over-completeness relation $\int d^2\alpha|\alpha\rangle\langle\alpha| = 1$. Dirac [2] was the first who decomposed a as $a = \sqrt{n}e^{i\phi}$, where n is the number operator $n = a^\dagger a$ and $e^{i\phi}$ is the Dirac phase operator. Later, Susskind and Glogower (SG) [3] improved it as

$$a = \sqrt{n+1}e^{i\phi}. \tag{1}$$

Equation (1) can be considered as the polar decomposition of a. However, the SG phase operator is not unitary, which implies that there does not exist a proper phase operator ϕ. To overcome this difficulty, in [4] using an eight-port homodyne detection scheme, Noh, Fougeres and Mandel (NFM) proposed an operational quantum phase measurement to introduce Hermitian phase operators. Later, Freyberger *et al* [5] pointed out that, in the limit of a strong local oscillator, the simultaneous measurable NFM phase operator pair can be defined. Meanwhile, Hradil [6] summarized NFM phase measurement and introduced a two-mode nonlinear phase operator as

$$\left(\frac{a+b^\dagger}{a^\dagger+b}\right)^{\frac{1}{2}} = e^{i\theta} \tag{2}$$

which is obviously unitary, noting $[a+b^\dagger, a^\dagger+b] = 0$, so they can reside in the same square root. In this work we generalize (1) to the two-mode case by introducing the following operator:

$$A = \sqrt{D+1}\left(\frac{a+b^\dagger}{a^\dagger+b}\right)^{\frac{1}{2}} \qquad A^\dagger = \left(\frac{a^\dagger+b}{a+b^\dagger}\right)^{\frac{1}{2}}\sqrt{D+1} \tag{3}$$

where $D \equiv a^\dagger a - b^\dagger b$ is the photon number difference between the two modes. (As one can see later we need not worry about whether the argument in the square root of (3) is less than zero.) By comparing equations (1) and (3) we see the correspondence $\sqrt{n+1} \to \sqrt{D+1}$, and $e^{i\phi} \to e^{i\theta}$. Since $e^{i\theta}$ is a nonlinear operator, the generalization form (1)–(3) is non-trivial. Due to the relations

$$[D, a+b^\dagger] = -(a+b^\dagger) \qquad [D, a^\dagger + b] = -(a^\dagger + b) \qquad [D, (a+b^\dagger)(a^\dagger + b)] = 0 \tag{4}$$

which indicate that

$$[D, e^{i\theta}] = -e^{i\theta} \qquad [D, e^{-i\theta}] = e^{-i\theta} \tag{5}$$

we have the familiar commutation relation

$$[A, A^\dagger] = 1. \tag{6}$$

The aim of this work is to search for the eigenvector $|\beta\rangle$ of the operator A

$$A|\beta\rangle = \beta|\beta\rangle \qquad \beta = |\beta| e^{i\varphi} \tag{7}$$

which we call the 'number-difference-phase coherent state analogue' in two-mode Fock space. In section 2, based on the result of [7] we construct $|\beta\rangle$. Then in section 3 we discuss its properties, especially its over-completeness relation.

2. The construction of $|\beta\rangle$

Recall that in [7] Fan and Zou have set up a new quantum mechanical representation denoted by $|q, r\rangle$, which is the common eigenvector of D and $(a+b^\dagger)(a^\dagger + b)$, satisfying

$$D|q,r\rangle = q|q,r\rangle \qquad (a+b^\dagger)(a^\dagger+b)|q,r\rangle = r^2|q,r\rangle \tag{8}$$

where q is an integer. The explicit expression of $|q, r\rangle$ in two-mode Fock space has also been derived:

$$|q,r\rangle = \exp\left[-\frac{1}{2}r^2 - a^\dagger b^\dagger\right] \sum_{n=\max(0,-q)}^{\infty} \frac{r^{2n+q}}{\sqrt{n!(n+q)!}} |n+q, n\rangle \tag{9}$$

$|q, r\rangle$ is proved to be complete:

$$\sum_{q=-\infty}^{\infty} \int_0^\infty dr^2 |q,r\rangle\langle q,r| = 1 \tag{10}$$

and orthonormal:

$$\langle q', r'|q, r\rangle = \frac{1}{2r'} \delta_{q,q'} \delta(r' - r). \tag{11}$$

It is also shown in [7] that the lowering and ascending properties of $e^{i\theta}$ and $e^{-i\theta}$ acting on $|q, r\rangle$ are satisfied:

$$e^{i\theta}|q,r\rangle = |q-1,r\rangle \qquad e^{i\theta}|q,r\rangle = |q+1,r\rangle. \tag{12}$$

Using the completeness relation of $|q, r\rangle$ in equation (10) the state $|\beta\rangle$ can be expanded as

$$|\beta\rangle = \sum_{q=-\infty}^{\infty} \int_0^\infty dr^2\, C_{q,r} |q,r\rangle \tag{13}$$

where $C_{q,r} = \langle q, r|\beta\rangle$. Obviously,

$$A|q,r\rangle = \sqrt{q}|q-1,r\rangle \tag{14}$$

which resembles the single-mode case $a|n\rangle = \sqrt{n}|n-1\rangle$, where $|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$ is the number state. However, $\sqrt{q}$ could be a pure imaginary number as q can take negative integer values, and there is no lower bound for $|q, r\rangle$. Substituting equation (14) into the eigenvector equation (7) we have

$$A|\beta\rangle = \sum_{q=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}r^2\, C_{q,r}|q-1, r\rangle\sqrt{q} = \beta \sum_{q=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}r^2\, C_{q,r}|q, r\rangle. \quad (15)$$

Employing the orthonormal property of $|q, r\rangle$ yields the recurrence relation of $C_{q,r}$, i.e.,

$$\sqrt{q}C_{q,r} = \beta C_{q-1,r} \quad (16)$$

which implies that $C_{q,r}$ can be decomposed as

$$C_{q,r} = C_q f(r). \quad (17)$$

Here $f(r)$ is a function of r (which will be determined briefly later) whereas C_q is not. In particular, $\beta C_{-1,r}|-1, r\rangle = 0$. Therefore, combining equations (16) and (17) gives the expression for C_q:

$$C_q = \begin{cases} \dfrac{\beta^q}{\sqrt{q!}}C_0 & \text{for} \quad q \geqslant 0 \\ 0 & \text{for} \quad q < 0 \end{cases} \quad (18)$$

where C_0 is a normalization constant. So we get from equations (17) and (18) that

$$C_{q,r} = \begin{cases} \dfrac{\beta^q}{\sqrt{q!}}f(r) & \text{for} \quad q \geqslant 0 \\ 0 & \text{for} \quad q < 0 \end{cases} \quad (19)$$

where we have absorbed C_0 into $f(r)$. Equation (19), together with (13), gives the expression of $|\beta\rangle$ in the $|q, r\rangle$ basis,

$$|\beta\rangle = \sum_{q=0}^{\infty} \int_0^{\infty} \mathrm{d}r^2 \frac{\beta^q}{\sqrt{q!}} f(r)|q, r\rangle. \quad (20)$$

Then only those states $|q, r\rangle$ with $q \geqslant 0$ contribute to $|\beta\rangle$, which is why we need not worry about the argument in the square root of (3) being negative. Now recall that $A^{\dagger}|q, r\rangle = \mathrm{e}^{-\mathrm{i}\theta}\sqrt{D+1}|q, r\rangle = \sqrt{q+1}|q+1, r\rangle$, which implies that, for $q \geqslant 0$, the following expression is valid:

$$|q, r\rangle = \frac{A^{\dagger q}}{\sqrt{q!}}|0, r\rangle. \quad (21)$$

This equation looks like the well known relation $a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle$. From equations (20) and (21) we obtain

$$|\beta\rangle = \sum_{q=0}^{\infty} \int_0^{\infty} \mathrm{d}r^2 \frac{\beta^q}{\sqrt{q!}} f(r) \frac{A^{\dagger q}}{\sqrt{q!}}|0, r\rangle = \mathrm{e}^{\beta A^{\dagger}} \int_0^{\infty} \mathrm{d}r^2\, f(r)|0, r\rangle. \quad (22)$$

To determine the normalization constant for $|\beta\rangle$ with respect to β we calculate

$$\begin{aligned} \langle\beta|\beta\rangle &= \int_0^{\infty}\int_0^{\infty} \mathrm{d}r^2\, \mathrm{d}r'^2 \langle 0, r'|\mathrm{e}^{\beta^* A}\mathrm{e}^{\beta A^{\dagger}}|0, r\rangle f(r) f(r') \\ &= \mathrm{e}^{|\beta|^2} \int_0^{\infty}\int_0^{\infty} \mathrm{d}r^2\, \mathrm{d}r'^2 \frac{1}{2r'}\delta(r'-r) f(r) f(r') \\ &= \mathrm{e}^{|\beta|^2} \int_0^{\infty} \mathrm{d}r^2\, f^2(r). \end{aligned} \quad (23)$$

Here $A|0,r\rangle = 0$, as well as the Baker–Hausdoff formula $\mathrm{e}^{\beta^* A}\mathrm{e}^{\beta A^\dagger} = \mathrm{e}^{|\beta|^2}\mathrm{e}^{\beta A^\dagger}\mathrm{e}^{\beta^* A}$, is used. Therefore, if we impose the condition for $f(r)$ which should satisfy

$$\int_0^\infty \mathrm{d}r^2\, f^2(r) = 1 \tag{24}$$

for instance, $f(r)$ can be $\mathrm{e}^{-\frac{1}{2}r^2}$, then the normalized eigenstate $|\beta\rangle$ is

$$|\beta\rangle = \mathrm{e}^{-\frac{|\beta|^2}{2}+\beta A^\dagger} \int_0^\infty \mathrm{d}r^2\, f(r)|0,r\rangle. \tag{25}$$

Thus we see that $|\beta\rangle$ is based on a superposition of the states $|0,r\rangle$ on the radius direction, provided that $f(r)$ satisfies equation (24). It must be emphasized that $|\beta\rangle$ is different from the usual two-mode coherent state, as is obvious.

3. The properties of $|\beta\rangle$

Further, let

$$\int_0^\infty \mathrm{d}r^2\, f(r)|0,r\rangle = |0\rangle\rangle \tag{26}$$

then it is obviously seen that $\langle\langle 0 \mid 0\rangle\rangle = 1$, $A|0\rangle\rangle = 0$ and

$$|\beta\rangle = \mathrm{e}^{-\frac{|\beta|^2}{2}+\beta A^\dagger}|0\rangle\rangle \tag{27}$$

from which then follows the non-orthogonal overlap:

$$\langle\beta'|\beta\rangle = \exp\left[-\frac{|\beta|^2+|\beta'|^2}{2} + \beta'^*\beta\right] \tag{28}$$

which is a distinguished property of the coherent state. Now, defining the un-normalized state $\|\beta\rangle = \mathrm{e}^{\beta A^\dagger}|0\rangle\rangle$, from equation (28) we have $\langle\beta'\|\beta\rangle = \mathrm{e}^{\beta'^*\beta}$. Therefore:

$$\langle\beta'\|\partial/\partial\beta\rangle\langle\beta^*\||_{\beta=0}\|\beta''\rangle = \mathrm{e}^{\beta'^*\frac{\partial}{\partial\beta}}\mathrm{e}^{\beta\beta''}|_{\beta=0} = \mathrm{e}^{\beta'^*\beta''}. \tag{29}$$

Since β' and β'' are arbitrarily taken, equation (29) means

$$\|\partial/\partial\beta\rangle\langle\beta^*\||_{\beta=0} = 1. \tag{30}$$

It then follows that

$$\|\partial/\partial\beta\rangle\langle\beta^*\||_{\beta=0} = \mathrm{e}^{\frac{\partial}{\partial\beta}A^\dagger}|0\rangle\rangle\langle\langle 0|\mathrm{e}^{\beta^* A}|_{\beta=0} = 1. \tag{31}$$

Now we introduce the normal product concept for A and $A^\dagger$, denoted as $\genfrac{}{}{0pt}{}{\circ}{\circ}\ \genfrac{}{}{0pt}{}{\circ}{\circ}$, and assume the normally ordered form of $|0\rangle\rangle\langle\langle 0|$ to be $\genfrac{}{}{0pt}{}{\circ}{\circ} W \genfrac{}{}{0pt}{}{\circ}{\circ}$, where W is to be determined. Then equation (31) becomes

$$\begin{aligned}\|\partial/\partial\beta\rangle\langle\beta^*\|_{\beta=0} &= \mathrm{e}^{\frac{\partial}{\partial\beta}A^\dagger}\,\genfrac{}{}{0pt}{}{\circ}{\circ} W \genfrac{}{}{0pt}{}{\circ}{\circ}\,\mathrm{e}^{\beta^* A}|_{\beta=0} = \genfrac{}{}{0pt}{}{\circ}{\circ}\,\mathrm{e}^{\frac{\partial}{\partial\beta}A^\dagger} W \mathrm{e}^{\beta^* A}\,\genfrac{}{}{0pt}{}{\circ}{\circ}|_{\beta=0}\\ &= \genfrac{}{}{0pt}{}{\circ}{\circ}\,\mathrm{e}^{A^\dagger A} W\,\genfrac{}{}{0pt}{}{\circ}{\circ} = 1.\end{aligned} \tag{32}$$

Notice that the operators A and $A^\dagger$ are permuted within the normal ordering symbol $\genfrac{}{}{0pt}{}{\circ}{\circ}\ \genfrac{}{}{0pt}{}{\circ}{\circ}$ [8], so

$$|0\rangle\rangle\langle\langle 0| = \genfrac{}{}{0pt}{}{\circ}{\circ} W \genfrac{}{}{0pt}{}{\circ}{\circ} = \genfrac{}{}{0pt}{}{\circ}{\circ}\,\mathrm{e}^{-A^\dagger A}\,\genfrac{}{}{0pt}{}{\circ}{\circ}. \tag{33}$$

Therefore we can show that

$$\begin{aligned}\int \frac{d^2\beta}{\pi}|\beta\rangle\langle\beta| &= \int \frac{|\beta| d|\beta|}{\pi} e^{-|\beta|^2} \int d\varphi \exp[A^\dagger|\beta|e^{i\varphi}]|0\rangle\rangle\langle\langle 0|\exp[A|\beta|e^{-i\varphi}] \\ &= \sum_{n=0}^{\infty} \frac{A^{\dagger n}}{\sqrt{n!}}|0\rangle\rangle\langle\langle 0|\frac{A^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{A^{\dagger n}}{\sqrt{n!}} {}^{\circ}_{\circ} e^{-A^\dagger A} {}^{\circ}_{\circ} \frac{A^n}{\sqrt{n!}} \\ &= {}^{\circ}_{\circ} e^{A^\dagger A - A^\dagger A} {}^{\circ}_{\circ} = 1 \end{aligned} \tag{34}$$

which indicates the over-completeness relation of $|\beta\rangle$. If we use the technique of integration within an ordered product ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ of operators [8] we can simplify (34) to

$$\int \frac{d^2\beta}{\pi}|\beta\rangle\langle\beta| = \int \frac{d^2\beta}{\pi} {}^{\circ}_{\circ} e^{-|\beta|^2+\beta A^\dagger+\beta^* A - AA} {}^{\circ}_{\circ} = 1. \tag{35}$$

4. $|\beta\rangle$ as a minimum uncertainty state

As the usual coherent state makes the coordinate-momentum uncertainty relation a minimum, we explore the corresponding property for $|\beta\rangle$. We introduce the Hermitian operators (new quadratures)

$$X = \frac{1}{\sqrt{2}}(A + A^\dagger) \qquad P = \frac{1}{i\sqrt{2}}(A - A^\dagger) \tag{36}$$

whose commutation relation is $[X, P] = i$, which implies the uncertainty relation $\Delta X \Delta P \geqslant \frac{1}{2}$. In $|\beta\rangle$ we have the following expectations:

$$\begin{aligned} &\langle\beta|X|\beta\rangle = \frac{1}{\sqrt{2}}(\beta + \beta^*) \qquad \langle\beta|P|\beta\rangle = \frac{1}{i\sqrt{2}}(\beta - \beta^*) \\ &\langle\beta|X^2|\beta\rangle = \tfrac{1}{2}(\beta^2 + \beta^{*2} + 2|\beta|^2 + 1) \\ &\langle\beta|P^2|\beta\rangle = -\tfrac{1}{2}(\beta^2 + \beta^{*2} - 2|\beta|^2 - 1) \end{aligned} \tag{37}$$

so that

$$(\Delta X)^2 = \langle\beta|X^2|\beta\rangle - \langle\beta|X|\beta\rangle^2 = \tfrac{1}{2} = (\Delta P)^2 \tag{38}$$

which implies that, for $|\beta\rangle$, we can state

$$\Delta X \Delta P = \tfrac{1}{2}. \tag{39}$$

Thus $|\beta\rangle$ makes the uncertainty relation minimum.

From the eight-port homodyne experiment we know both the operational phase $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and the two-mode photon number difference D can be measured, as they are both observables. So the operators A and $A^\dagger$ can be indirectly known. Thus the introduction of operators A and $A^\dagger$ is natural, and they may have a physical meaning in their own right once the experimentalists can figure out how to measure them directly.

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